Answers to the Problems in Chapter 4

Problem 4.1.

The rectangular array of $m_{d1} + m_{d2}$ values is:

		m_{d1}				
		+2	+1	0	–1	–2
	+3/2	+7/2	+5/2	+3/2	+1/2	–1/2
m_{d2}	+1/2	+5/2	+3/2	+1/2	–1/2	–3/2
	–1/2	+3/2	+1/2	–1/2	–3/2	–5/2
	–3/2	+1/2	–1/2	–3/2	–5/2	–7/2

Clearly, the first row and the last column are the m_d values for a state with d = 7/2. Similarly, the second row as far as the penultimate column, plus the remainder of that column are the m_d values of a state with d = 5/2. One further repeat of this procedure leaves the two m_d values of +1/2 and –1/2 in the last row.

Problem 4.2.

From Equation 4.1.2, the angular momentum of the two rotating masses of a diatomic molecule will be, $a = 2rmv$

If the time for one rotation is τ then, $2\pi r/\tau = v$

And because $\tau = 1/\nu_{rot}$, where ν_{rot} is the frequency of rotation we have:

$$a = 2rm\bullet 2\pi r\nu_{rot} = 4\pi r^2 m\nu_{rot}$$

For J = 1 $\quad a = \{J(J+1)\}^{½}h/2\pi = \sqrt{2}h/2\pi$

Therefore, $\quad \nu_{rot} = \sqrt{2}h/8\pi^2 r^2 m = h/4\sqrt{2}\pi^2 r^2 m$

$$= (6.62608\times10^{-34})/(5.65685\times9.86958\times0.36453\times26.56019\times10^{-47})$$

$$= 12.258\times10^{10}\ s^{-1}.$$

Problem 4.3.

$\|d, m_d\rangle$	$\hat{l}_+\|d, m_d\rangle/(h/2\pi)$	$\hat{l}_-\|d, m_d\rangle/(h/2\pi)$
$\|3,+3\rangle$	$\{12–12\}^{½} = 0$	$\{12–6\}^{½} = \{6\}^{½}\|3,+2\rangle$
$\|3,+2\rangle$	$\{12–6\}^{½} = \{6\}^{½}\|3,+3\rangle$	$\{12–2\}^{½} = \{10\}^{½}\|3,+1\rangle$
$\|3,+1\rangle$	$\{12–2\}^{½} = \{10\}^{½}\|3,+2\rangle$	$\{12–0\}^{½} = \{12\}^{½}\|3, 0\rangle$
$\|3, 0\rangle$	$\{12–0\}^{½} = \{12\}^{½}\|3,+1\rangle$	$\{12–0\}^{½} = \{12\}^{½}\|3,–1\rangle$
$\|3,–1\rangle$	$\{12–0\}^{½} = \{12\}^{½}\|3, 0\rangle$	$\{12–2\}^{½} = \{10\}^{½}\|3,–2\rangle$
$\|3,–2\rangle$	$\{12–2\}^{½} = \{10\}^{½}\|3,–1\rangle$	$\{12–6\}^{½} = \{6\}^{½}\|3,–3\rangle$
$\|3,–3\rangle$	$\{12–6\}^{½} = \{6\}^{½}\|3,–2\rangle$	$\{12–12\}^{½} = 0$

Problem 4.4.

Let the three protons be designated a, b and c. We first couple a and b. The possible combinations of $m_a + m_b = M_{ab}$ are:

m_a	m_b	M_{ab}	m_a	m_b	M_{ab}	m_a	m_b	M_{ab}
+½	+½	+1	+½	–½	0	–½	–½	–1
			–½	+½	0			

These combinations correspond to a state with spin $S_{ab} = 1$ and a state with $S_{ab} = 0$.
We now combine those states with that of the remaining proton to give the total M value.

For $S_{ab} = 1$

m_c	M_{ab}	M	m_c	M_{ab}	M	m_c	M_{ab}	M
+½	+1	+3/2	+½	0	+½	+½	–1	–½
–½	+1	+½	–½	0	–½	–½	–1	–3/2

For $S_{ab} = 0$

m_c	M_{ab}	M	m_c	M_{ab}	M
+½	0	+½	–½	0	–½

We find that the values of M correspond to a state having total spin S = 3/2 with four M_S components, a *quartet*, and two states having S = ½ each with two M_S components, two *doublets*. There are 8 components in total corresponding to the eight possible nuclear spin functions, 2×2×2.

As Box 4.2 shows, each of the three steps above is really an illustration of the Clebsch-Gordan series.

In the first we find S_{ab} values of ½+½ to |½–½|, i.e. 1 and 0.

In the second we find S values of 1+½ to |1–½|, i.e. 3/2 and ½; the quartet and one doublet.

In the third step we find S values of 0+½ to |0–½|, i.e. ½; the second doublet.

Problem 4.5.

In the following the notation is simplified so that, for example,

|+½(1), –½(2), +½(3)> is written as |+ – +>

(a) It has already been shown that from

|+3/2,+3/2> = |+ + +>

by use of the lowering operators $\hat{S}_-$ and $\sum_i \hat{s}_-(i)$ we can obtain the result:

$$|3/2,+1/2\rangle = (1/\sqrt{3})\{|-++\rangle + |+-+\rangle + |++-\rangle\}$$

(b) Proceeding further in exactly the same way, we first operate with $\hat{S}_-$ on the left-hand side of the last equation and obtain:

$$\hat{S}_-|3/2,-1/2\rangle = 2|3/2,-1/2\rangle$$

Operating with $\sum_i \hat{s}_-(i)$ on the right-hand side of the same equation we obtain:

$$\sum_i \hat{s}_-(i)\,(1/\sqrt{3})\{|-++\rangle + |+-+\rangle + |++-\rangle\}$$

$$= (1/\sqrt{3})\{|--+\rangle + |-+-\rangle + |--+\rangle + |+--\rangle + |-+-\rangle + |+--\rangle\}$$

$$= (2/\sqrt{3})\{|--+\rangle + |-+-\rangle + |+--\rangle\}$$

Therefore

$$|3/2,-1/2\rangle = (1/\sqrt{3})\{|--+\rangle + |-+-\rangle + |+--\rangle\}$$

Similarly, we can show that $|3/2,-3/2\rangle = |---\rangle$

(c) Because the individual spin functions are normalised and orthogonal, i.e.

$$\langle +½|+½\rangle = \langle -½|-½\rangle = 1.0 \quad \text{and} \quad \langle +½|-½\rangle = \langle -½|+½\rangle = 0.0$$

an integral of the form $\langle a\,b\,c|a'\,b'\,c'\rangle = \langle a|a'\rangle\bullet\langle b|b'\rangle\bullet\langle c|c'\rangle$ over the three spin functions is zero unless a = a', b = b' and c = c'.

$$|1/2,+1/2\rangle = (1/\sqrt{6})\{2|-++\rangle - 1|+-+\rangle - 1|++-\rangle\}$$

$$|1/2,+1/2\rangle^{\dagger} = (1/\sqrt{2})\{1|+-+\rangle - 1|++-\rangle\}$$

Normalisation

$$\langle 1/2,+1/2|1/2,+1/2\rangle = (1/6)\{4\langle -++|-++\rangle + 1\langle +-+|+-+\rangle + 1\langle ++-|++-\rangle\}$$

$$=(1/6)\{4 + 1 + 1\} = 1.0$$

$${}^{\dagger}\langle 1/2,+1/2|1/2,+1/2\rangle^{\dagger} = (1/2)\{1\langle +-+|+-+\rangle + 1\langle ++-|++-\rangle\}$$

$$=(1/2)\{1 + 1\} = 1.0$$

Orthogonality

$$\langle 1/2,+1/2|1/2,+1/2\rangle^{\dagger} = (1/\sqrt{[12]})\{-1\langle +-+|+-+\rangle + 1\langle ++-|++-\rangle\}$$

$$=(1/\sqrt{[12]})\{1 - 1\} = 0.0$$

To find $|1/2,-1/2\rangle$ we have:

$$\hat{S}_-|1/2,+1/2\rangle = |1/2,-1/2\rangle$$

and

$$\sum_i \hat{s}_-(i)\,(1/\sqrt{6})\{2|-++\rangle - 1|+-+\rangle - 1|++-\rangle\}$$

$$= (1/\sqrt{6})\{2|--+\rangle + 2|-+-\rangle - 1|--+\rangle - 1|+--\rangle - 1|-+-\rangle - 1|+--\rangle\}$$

$$= (1/\sqrt{6})\{1|--+\rangle + 1|-+-\rangle - 2|+--\rangle\}$$

Thus, $|1/2,-1/2\rangle = (1/\sqrt{6})\{1|--+\rangle + 1|-+-\rangle - 2|+--\rangle\}$

Note that the function is correctly normalised.